

Lecture 7

Today: * Rockafellar's theorem
+ The dual problem

From last time:

Theorem: If $\Gamma \subset X \times Y$ is c -cyclically
monotone, there exists a c -convex function
 $\phi: X \rightarrow \mathbb{R}$ s.t.
 $(x, y) \in \Gamma \Rightarrow y \in \partial_c \phi(x)$.

Proof (continued).

Last time we fixed (x_0, y_0) and
introduced the function $\phi: X \rightarrow \mathbb{R}$ by

$$\phi(x) = \sup \left\{ -c(x, y_0) + c(x_0, y_0) + \sum_{k=1}^n -c(x_{k-1}, y_k) + c(x_k, y_k) \right\}$$

$$\left| \left\{ (x_k, y_k) \right\}_{k=0}^{\infty} \subset \Gamma, \text{ and } \left. \begin{array}{l} x_n = x_*, y_n = y_* \end{array} \right\} \right\}$$

we noted that $\phi(x_*) \leq 0$ by c -cyclical monotonicity, and that in this case ϕ is well defined and inherits whatever modulus of continuity $x \mapsto c(x, y)$ has that is uniform in y .

What remains to show is the two key facts:

- ① $\phi(x)$ is c -convex
- ② If $(\hat{x}, \hat{y}) \in \Gamma$, then $\hat{y} \in \partial_c \phi(\hat{x})$

Let's show ② (for ①, see the exercise after the proof)

Fix $(\hat{x}, \hat{y}) \in \Gamma$. Then, $\forall \varepsilon > 0 \exists$
a sequence $\{(x_k^{(\varepsilon)}, y_k^{(\varepsilon)})\}_{k=0}^n$ s.t.

$$\phi(x) \geq -c(x_1, y_0^{(\varepsilon)}) + c(x_0^{(\varepsilon)}, y_0^{(\varepsilon)}) \\ + \left\{ \sum_{k=1}^n -c(x_{k-1}^{(\varepsilon)}, y_k^{(\varepsilon)}) + c(x_k^{(\varepsilon)}, y_k^{(\varepsilon)}) \right\} \\ \forall x$$

and

$$\phi(\hat{x}) - \varepsilon \leq -c(\hat{x}_1, y_0^{(\varepsilon)}) + c(x_0^{(\varepsilon)}, y_0^{(\varepsilon)}) \\ + \left\{ \sum_{k=1}^n -c(x_{k-1}^{(\varepsilon)}, y_k^{(\varepsilon)}) + c(x_k^{(\varepsilon)}, y_k^{(\varepsilon)}) \right\}$$

Since

$$\{(\hat{x}, \hat{y}), (x_0^{(\varepsilon)}, y_0^{(\varepsilon)}), \dots, (x_n^{(\varepsilon)}, y_n^{(\varepsilon)})\} \subset \Gamma$$

we have for all $x \in X$,

$$\begin{aligned} \phi(x) \geq & -C(x, \hat{y}) + C(\hat{x}, \hat{y}) \\ & - C(\hat{x}, y_0^{(\varepsilon)}) + C(x_0^{(\varepsilon)}, y_0^{(\varepsilon)}) \\ & + \sum_{k=1}^n -C(x_{k-1}^{(\varepsilon)}, y_k^{(\varepsilon)}) + C(x_k^{(\varepsilon)}, y_k^{(\varepsilon)}) \end{aligned}$$

write

$$\begin{aligned} \alpha^{(\varepsilon)} := & -C(\hat{x}, y_0^{(\varepsilon)}) + C(x_0^{(\varepsilon)}, y_0^{(\varepsilon)}) \\ & + \sum_{k=1}^n -C(x_{k-1}^{(\varepsilon)}, y_k^{(\varepsilon)}) + C(x_k^{(\varepsilon)}, y_k^{(\varepsilon)}) \end{aligned}$$

Then, $\forall x \in X$,

$$\begin{aligned} \phi(x) \geq & -C(x, \hat{y}) + C(\hat{x}, \hat{y}) \\ & + \alpha^{(\varepsilon)} \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ and using the definition of $\{ (x_k^{(\varepsilon)}, y_k^{(\varepsilon)}) \}_{k=0}^n$ we conclude that

$$\phi(x) \geq -C(x, \tilde{y}) + C(x, \hat{y}) + \phi(\hat{x})$$

$$\Rightarrow \hat{y} \in \partial_c \phi(\hat{x}).$$

For ②, see the following exercise

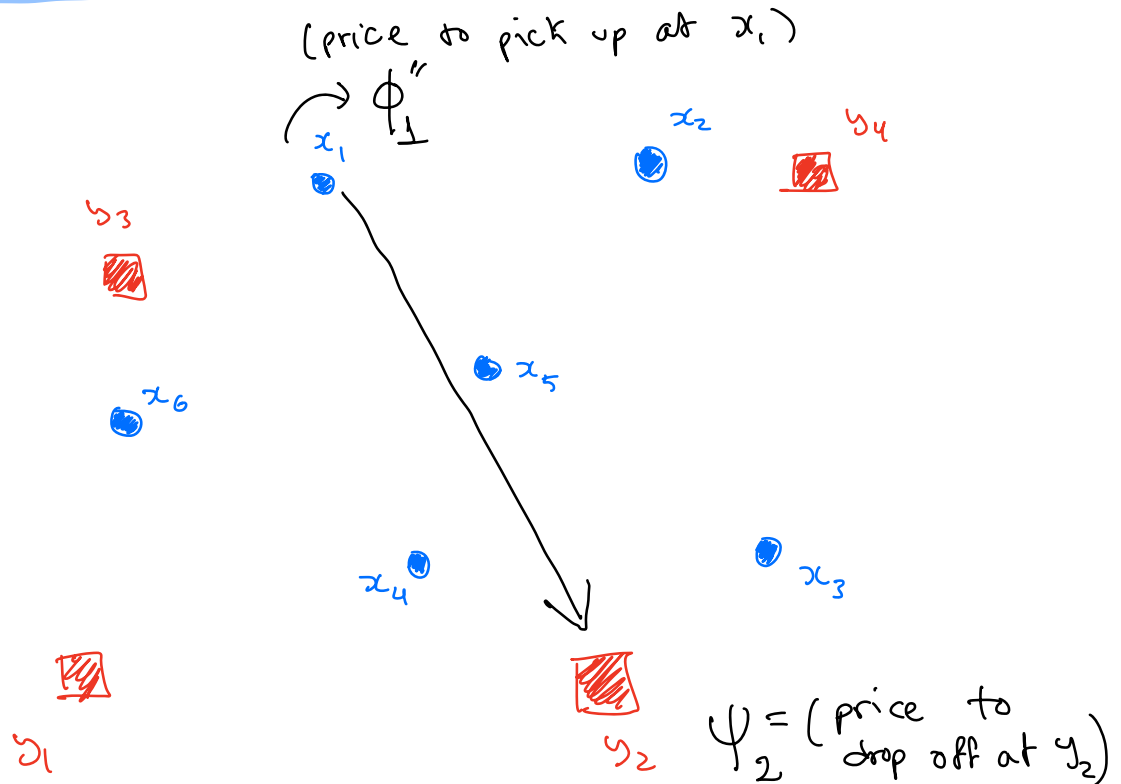


Exercise : Consider a family of c -convex functions $\{\phi_i\}_{i \in I}$ and define

$$\phi(x) = \sup_i \phi_i(x)$$

Show that if $\{\phi_i\}$ is equibounded and uniformly continuous then ϕ is c -convex.

The dual problem AKA the shippers' dilemma



Let us consider the situation we presented when we introduced the discrete Kantorovich problem.

Suppose you run a shipping company;

you charge an amount ϕ_i for the picking up of one unit of mass at a location x_i , and charge an amount ψ_j for putting a unit of mass at location y_j .

Assume the amount of mass (say, lumber) at x_i is f_i , and the demand at y_j is g_j , with $\sum f_i = \sum g_j$. Then, the total income to the shipper is

$$\sum_{i=1}^m \phi_i f_i + \sum_{j=1}^n \psi_j g_j$$

The shipper wants to make this as large as possible. However, the shipper has the contend with the constrain

$$\phi_i + \psi_j \leq c_{ij}$$

For, if the opposite inequality held, the lumber mill at x_i and factory y_j would rather do the transportation themselves, with a lower cost of c_{ij} .

Therefore, we have the following maximization problem

$$\text{maximize} \quad \sum_{i=1}^m \phi_i f_i + \sum_{j=1}^n \psi_j g_j$$

$$\text{subject to} \quad \phi_i + \psi_j \leq c_{ij}$$

This is the (discrete) dual problem to the Kantorovich problem.

The dual problem (general case)

Given $c: X \times Y \rightarrow \mathbb{R}$, continuous, and $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$ (X, Y assumed compact)
then, consider the following maximization problem

$$\max_{\phi, \psi} \int_X \phi d\mu + \int_Y \psi d\nu$$

$$\begin{aligned} \text{subject to} \quad & \phi(x) + \psi(y) \leq c(x, y) \\ & \phi \in L^1(\mu), \psi \in L^1(\nu) \end{aligned}$$

(The Dual Problem)